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Qualitative behaviour of viscoplastic solutions in the vicinity of maximum-friction surfaces

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Abstract The maximum-friction surface is a source of singular solution behaviour for several rate-independent plasticity models. Solutions based on conventional viscoplastic models do not show such behaviour. For a class of materials, there is a range of temperatures and/or strain rates where a necessity of the consideration of rate effects depends on the area of application of the final result. Hence, the same material under the same conditions can be represented by either rate-independent or rate-dependent models. In this case, a reasonable requirement is that viscous effects should not be very significant and, in particular, the qualitative behaviour of viscoplastic solutions should be similar to that of solutions based on rate-independent models. The present paper deals with this issue by means of the solution for simultaneous shearing and expansion of a hollow cylinder under plane-strain deformation. One of the goals of the paper is to show that there is a class of viscoplastic models satisfying the requirement formulated. The other goal is to find an asymptotic representation of the solution in the vicinity of the maximum-friction surface and compare it with the rigid perfectly plastic solution.

Keywords Asymptotic analysis · Friction · Singularity · Viscoplasticity

1 Introduction

The behaviour of rigid plastic solutions in the vicinity of maximum-friction surfaces depends on the constitutive equations chosen and can cause numerical difficulty. For pressure-independent material models, the maximum-friction law postulates that the friction stress is equal to the local shear yield stress. The simplest model of this class, the rigid perfectly plastic solid, leads to singular velocity fields in the vicinity of maximum-friction surfaces where some components of the strain-rate tensor and the equivalent strain-rate (the second invariant of the strain-rate

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tensor) approach infinity [1]. When finite-element methods are used, such behaviour of the velocity field requires, in general, special shape functions in finite elements adjacent to the friction surface. The same asymptotic behaviour of the velocity field in the vicinity of maximum-friction surfaces occurs when the double-shearing model [2] is adopted [3,4]. This model is used for describing the deformation of granular materials. It is interesting to mention that other models, also used for such materials, can lead to qualitatively different behaviour of the velocity field near the maximum-friction surface [5–8]. Of several models of pressure-dependent plasticity considered in these papers, the double-slip and rotation model [9], in addition to the double-shearing model, only leads to the same solution behaviour near the maximum-friction surfaces as the model of rigid perfectly plastic solids.

Viscoplastic models can be divided into two groups. An important feature of conventional models, such as Bingham solids, is that $\sigma_Y \to \infty$ as $\xi_{eq} \to \infty$ where σ_Y is the yield stress in tension and ξ_{eq} is the equivalent strain rate. For such models, the maximum-friction law is equivalent to the regime of sticking at the friction surface [10], and such a regime of friction leads to a boundary layer near the friction surface but all components of the strain-rate tensor are bounded [11]. However, the assumption that $\sigma_Y \to \sigma_s < \infty$ as $\xi_{eq} \to \infty$ (models of the second group) can significantly change the solution behaviour near the maximum-friction surfaces. In particular, it has been demonstrated in [12] by means of the exact solution to a particular plane-strain problem that the velocity field can be singular and its behaviour can be the same as in the case of rigid perfectly plastic solutions. Moreover, it has been shown that any set of experimental data can be approximated by a model of the second group with the same accuracy as by a conventional model. A disadvantage of the solution considered in [12] is that the normal strain rates vanish at the friction surface (in a local Cartesian coordinate system one axis of which is orthogonal to the friction surface). This may have a significant effect on the solution behaviour at the friction surface [13]. Therefore, a new problem free of the aforementioned disadvantage is formulated and solved in the present paper. Then, an asymptotic analysis of the solution is carried out to determine the behaviour of the velocity field in the vicinity of the friction surface.

The research is motivated by both fundamental and applied aspects. On the fundamental side, it is of interest to understand the dependence of the relation between σ_Y and ξ_{eq} on the asymptotic behaviour of velocity fields in the vicinity of maximum-friction surfaces. On the applied side, a novel theory for describing the evolution of material properties can be developed based on the strain-rate intensity factor [14]. This factor has been introduced for rigid perfectly plastic solids in [13] and is the coefficient of the leading singular term in an asymptotic expansion of the equivalent strain rate in the vicinity of maximum-friction surfaces. Since the equivalent strain rate is involved in many evolution equations for material properties, the singular behaviour of the velocity field predicts a very large gradient of these properties near friction surfaces, which is in qualitative agreement with experiment (for example, [15]). However, most of such experimental results have been obtained for deformation processes at elevated temperatures where the effect of viscosity should be taken into account. Therefore, viscoplastic models that permit singular behaviour of the velocity fields may play an important role in applications of the aforementioned theory based on the strain-rate intensity factor to practical problems.

2 Material model

In cylindrical coordinates $r\theta z$, the constitutive behaviour of isotropic visco-rigid/plastic incompressible material under plane-strain deformation (ξ_{zz} is one of the principal strain rates and $\xi_{zz} = 0$) may be described by the following equations (this system of coordinates has been chosen for further convenience)

$$s_{r\theta} = \left(\sigma_Y/\sqrt{3}\right) \sin 2\varphi, \quad s_{rr} = -s_{\theta\theta} = -\left(\sigma_Y/\sqrt{3}\right) \cos 2\varphi, \tag{1}$$

$$\xi_{r\theta} = \xi_{\theta\theta} \tan 2\varphi, \tag{2}$$

$$\xi_{rr} + \xi_{\theta\theta} = 0, \tag{3}$$

where s_{rr}, $s_{\theta\theta}$ and $s_{r\theta}$ are the deviatoric portions of the stress tensor, ξ_{rr}, $\xi_{\theta\theta}$ and $\xi_{r\theta}$ are the components of the strain-rate tensor, and φ is an auxiliary function which may depend on r and θ, $0 \le \varphi \le \pi/4$. The yield stress in tension, σ_Y, is a prescribed function of the equivalent strain rate defined by

$$\xi_{eq} = \sqrt{(2/3)\,\xi_{ij}\xi_{ij}}. \tag{4}$$

The equivalent stress is defined by

$$\sigma_{eq} = \sqrt{(3/2)\,s_{ij}s_{ij}}. \tag{5}$$

The system of equations (1) to (3) is equivalent to Mises-type yield criterion, $\sigma_{eq} = \sigma_Y$, and its associated flow rule. Of special interest for the purpose of the present paper are constitutive laws in which $\sigma_Y \to \sigma_s < \infty$ as $\xi_{eq} \to \infty$ where σ_s can be named the saturation stress by analogy to a somewhat similar quantity in strain-hardening materials [16]. The maximum-friction law for such materials is given by

$$\tau_f = \sigma_s/\sqrt{3}, \tag{6}$$

at sliding. Here τ_f is the friction stress. In the case of the simplest visco-rigid/plastic models, the yield stress in tension depends on the equivalent strain rate only. Then, without the loss of generality, the relation between σ_Y and ξ_{eq} can be written in the form

$$\sigma_Y = \sigma_s \Phi\left(\xi_{eq}/\xi_0\right) = \sigma_s \Phi\left(\eta\right), \tag{7}$$

where $\eta = \xi_{eq}/\xi_0$ and $\xi_0 = $ constant. It is assumed that the function $\Phi(\eta)$ is given and satisfies the following conditions:

$$\Phi(0) = \phi_0 \equiv \frac{\sigma_0}{\sigma_s} < 1, \quad \lim_{\eta\to\infty} \Phi(\eta) = 1, \quad \Phi'(\eta) \equiv \frac{d\Phi}{d\eta} > 0, \quad \eta \in (0, \infty) \tag{8}$$

with σ_0 being the yield stress in tension at $\xi_{eq} = 0$.

The system of equations (1) to (3) should be supplemented with the equilibrium equations. For the state of stress independent of θ these equations have the form

$$\frac{d\sigma}{dr} + \frac{ds_{rr}}{dr} + \frac{s_{rr} - s_{\theta\theta}}{r} = 0, \qquad \frac{ds_{r\theta}}{dr} + \frac{2s_{r\theta}}{r} = 0, \tag{9}$$

where σ is the hydrostatic stress.

3 Formulation of the boundary-value problem

Consider an infinite circular hollow cylinder of internal radius a and external radius b subject to a system of loading consisting of normal and tangential stresses on its internal and external radii (Fig. 1). Due to this system of loading the cylinder is both expanded and twisted. It is convenient to introduce a cylindrical polar coordinate system with its z-axis coinciding with the axis of symmetry of the cylinder. The state of stress and strain-rate are plane with the plane of flow being orthogonal to the z-axis. The rate of expansion of the internal radius will be denoted by $\dot{a} \equiv da/dt$. The external radius is fixed against rotation. Therefore, the velocity boundary conditions are

$$u = \dot{a}, \quad \text{at } r = a \tag{10}$$

and

$$v = 0, \quad \text{at } r = b. \tag{11}$$

Here u and v are the radial and circumferential velocities, respectively. One of the stress boundary conditions is

$$\sigma_{rr} = -\sigma_s p_a < 0, \tag{12}$$

at $r = a$. Here $p_a > 0$ is given, but its value is not essential for understanding the general features of the solution. It is, however, assumed, for consistency of the problem definition, that the value of p_a is such that $\sigma_{rr} < 0$ at $r = b$. The final boundary condition is the friction law at $r = a$. Its specific form depends on the regime of friction, namely

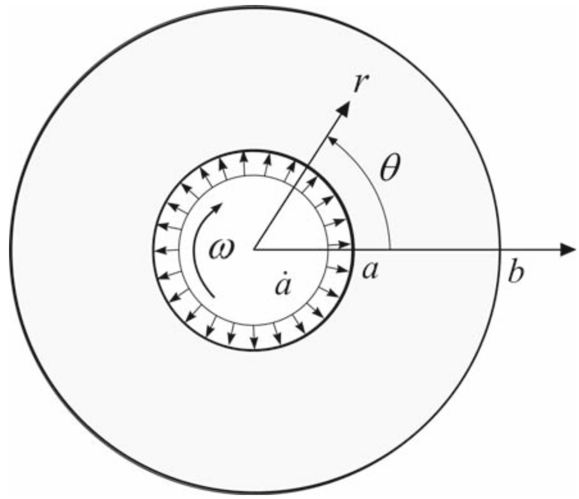

Fig. 1 Configuration of the process of deformation

whether the material is sliding or sticking at the inner boundary. For sliding the value of the shear stress at $r = a$ is given by (6). For sticking the circumferential velocity is prescribed:

$$v = -u_t, \tag{13}$$

at $r = a$. In the case of expansion of the cylinder $\xi_{\theta\theta} > 0$. Therefore, it follows from (2) that $\xi_{r\theta} > 0$ since it has been chosen that $0 \leq \varphi \leq \pi/4$. The inequality $\xi_{r\theta} > 0$ requires $u_t > 0$. The quantity u_t can be regarded as the circumferential velocity of points of the tool surface assuming that the tool is an expanding and rotating rod inserted into the hole of the cylinder. The rate of rod expansion is indeed equal to $\dot{a}$.

The model under consideration is history-independent because dynamic effects are not taken into account in the present analysis. Hence, it is sufficient to consider the instantaneous solution, i.e., for a specific geometry, to understand its general features.

Obviously, the problem formulated is not feasible for practical realization. However, it is a remarkable problem for a theoretical study of solution behaviour in the vicinity of frictional interfaces since its solution can be obtained in a closed form in terms of ordinary integrals without any approximation of the boundary conditions as in the case of more realistic problems such as compression of a block between two parallel, rough plates or flow of plastic material through converging channels. The present problem has already been successfully used for understanding some general features of solution behaviour for other material models [7].

4 Solution

The state of strain is plane and the solution is independent of θ. Therefore, equations (9) are valid and the normal strain rates are expressed in terms of the radial velocity as $\xi_{rr} = \mathrm{d}u/\mathrm{d}r$ and $\xi_{\theta\theta} = u/r$. Then, the radial velocity is determined from the incompressibility equation (3), with the use of the boundary condition (10), and the stress $s_{r\theta}$ from the equilibrium equation $(9)^2$ and $(1)^1$ in the form

$$u = \dot{a}/\rho, \tag{14}$$

$$s_{r\theta} = \frac{\sigma_Y}{\sqrt{3}} \sin 2\varphi = \frac{\sigma_s \tau}{\sqrt{3}\rho^2}, \tag{15}$$

where $\tau > 0$ is an arbitrary function of a (constant of integration) and $\rho = r/a$. From (7) Eq. 15 can be transformed to

$$\Phi(\eta) \sin 2\varphi = \frac{\tau}{\rho^2}. \tag{16}$$

The equilibrium equation (9)[1], by use of (7) and (12), determines the hydrostatic stress in the form

$$\frac{\sigma}{\sigma_s} = \frac{2}{\sqrt{3}} \int_1^\rho \frac{\Phi(\eta)\cos 2\varphi}{\rho} \mathrm{d}\rho + \frac{\Phi(\eta)\cos 2\varphi}{\sqrt{3}} - p_a. \tag{17}$$

The equivalent strain rate can be found by means of (2), (4) and (14) as

$$\xi_{\mathrm{eq}} = \frac{2}{\sqrt{3}} \frac{\dot{a}}{a\rho^2 \cos 2\varphi}. \tag{18}$$

Substitution of (18) in (16) gives

$$\Phi\left(\frac{2}{\sqrt{3}} \frac{\dot{a}}{a\xi_0\rho^2 \cos 2\varphi}\right) \sin 2\varphi = \frac{\tau}{\rho^2}. \tag{19}$$

If τ were known, this equation would determine φ as a function of ρ in implicit form with a being a parameter. Finally, integration of (2) with the use of the boundary condition (11) and the relation $2\xi_{r\theta} = \mathrm{d}v/\mathrm{d}r - v/r$ gives

$$\frac{v}{\dot{a}} = 2\rho \int_{b/a}^\rho \frac{\tan 2\varphi}{\rho^3} \mathrm{d}\rho. \tag{20}$$

The solution thus obtained satisfies Eqs. (1)–(3), (9) and the boundary conditions (10)–(12). It contains an undetermined function $\tau(a)$ which must be found from the friction boundary condition.

Consider the solution under sticking conditions. Combining (13) and (20) leads to

$$\frac{u_t}{\dot{a}} = 2 \int_1^{b/a} \frac{\tan 2\varphi}{\rho^3} \mathrm{d}\rho. \tag{21}$$

The equation for τ is obtained by excluding φ in (21) by means of (19). This equation may or may not have a solution depending on b/a, $u_t/\dot{a}$ and the form of function $\Phi(\eta)$.

If (21) has no solution, it is necessary to consider the regime of sliding for $\rho = 1$. In this case, it follows from (6), where τ_f should be replaced with $s_{r\theta}$, and (15) that a necessary condition is

$$\tau = 1. \tag{22}$$

Therefore, Eq. (19) can be rewritten in the form

$$\Phi\left(\frac{E_0}{\rho^2 \cos 2\varphi}\right) \sin 2\varphi = \frac{1}{\rho^2}, \tag{23}$$

with E_0 being the dimensionless parameter, $E_0 = 2\dot{a}/(\sqrt{3}a\xi_0)$. Substitution of (6) in (5) shows that $\sigma_{\mathrm{eq}} = \sigma_s$ and $s_{rr} = s_{\theta\theta} = 0$ at the friction surface. Therefore, it follows from (1) that in the case of sliding

$$\varphi \to \pi/4 \quad \text{as } \rho \to 1, \tag{24}$$

and, then, from (18) that

$$\xi_{\mathrm{eq}} \to \infty \quad \text{as } \rho \to 1. \tag{25}$$

5 Asymptotic analysis

Equation (22) provides a necessary condition for the regime of sliding to occur. However, in general, it is not a sufficient condition. In other words, the regime of sticking may occur if (22) is satisfied. Therefore, the goals of the asymptotic analysis completed in this section are to determine the precise conditions for each friction regime (sticking and sliding) to occur and to derive the asymptotic representation of the equivalent strain rate in the vicinity of the maximum-friction surface. It is assumed in this section that $E_0 \neq 0$ and $E_0 < \infty$.

Since $\varphi \in (0, \pi/4)$, this quantity can be eliminated between (16) and (19) to arrive at

$$\rho^4 = \frac{\tau^2}{\Phi^2(\eta)} + \frac{E_0^2}{\eta^2}, \qquad 1 \le \rho \le \frac{b}{a}, \tag{26}$$

This equation determines η as a function of ρ and τ in implicit form. In particular, it follows from (26) that:

$$\left(\frac{\tau^2\Phi'}{\Phi^3} + \frac{E_0^2}{\eta^3}\right)\mathrm{d}\eta = \frac{\tau}{\Phi^2}\mathrm{d}\tau - 2\rho^3\mathrm{d}\rho. \tag{27}$$

Because of the restrictions imposed on the function $\Phi(\eta)$ and specified in (8), the coefficient of $\mathrm{d}\eta$ in (27) is positive. Therefore, it follows from (27) that

$$\frac{\partial}{\partial\rho}\eta(\rho,\tau) < 0, \qquad \frac{\partial}{\partial\tau}\eta(\rho,\tau) > 0 \tag{28}$$

at $\tau > 0$ and $1 < \rho < b/a$. A combination of (8) and (28) allows one to conclude that the solution to (26) exists if and only if

$$0 \le \tau \le 1. \tag{29}$$

This conclusion immediately follows from (1), (7), (8) and (15) as well.

Excluding φ in (20) by means of (16) and (19) it is possible to get at $\rho = b/a$

$$\frac{U(\tau)}{\dot{a}} = \varpi(\tau) = \frac{2}{E_0}\int_1^{b/a} \frac{\sqrt{\rho^4\eta^2 - E_0^2}\mathrm{d}\rho}{\rho^3}. \tag{30}$$

Here η is understood as a function of ρ and τ due to (26) and U is the value of the circumferential velocity at $\rho = b/a$ and any assumed value of τ in the interval (29). It follows from (28) that the integrand in (30) is a monotonically increasing function of τ. As a result, the introduced parameter $\varpi(\tau)$ monotonically increases from 0 to ϖ_{cr} determined from (30) as

$$\varpi_{\mathrm{cr}} = \lim_{\tau\to 1}\varpi(\tau) = \frac{2}{E_0}\int_1^{b/a} \chi(\rho)\frac{\sqrt{\rho^4 - E_0^2/\chi^2(\rho)}\mathrm{d}\rho}{\rho^3}, \tag{31}$$

where $\chi(\rho)$ is defined from (26) in the following implicit form by putting $\tau = 1$ and replacing η with χ

$$\rho^4 = \frac{1}{\Phi^2(\chi)} + \frac{E_0^2}{\chi^2}. \tag{32}$$

The corresponding value of U is the maximum possible circumferential velocity at $\rho = b/a$, $U_{\max} = \dot{a}\varpi_{\mathrm{cr}}$.

By definition, the regime of sticking at $r = a$ occurs if it is possible to find such a value of τ that $U = u_t$. Therefore, the condition $\varpi_{\mathrm{cr}} = \infty$ is equivalent to the statement that the regime of sticking occurs at any value of u_t. At $\varpi_{\mathrm{cr}} < \infty$, the two friction regimes (sticking and sliding) are separated by the following inequalities

- sticking occurs at the surface $r = a$ if $u_t < U_{\max}$,
- sliding occurs at the surface $r = a$ if $u_t > U_{\max}$.

The special case $u_t = U_{\max}$ corresponds to the transition from one of the regimes to the other and can be regarded as a special case of sticking since there is no velocity jump at the interface. In this special case and in the case of sliding $\chi(\rho) \equiv \eta(\rho, 1)$. It is obvious from (31) that the value of ϖ_{cr} depends on the parameters involved in its definition, b/a and E_0, and the function $\Phi(\eta)$. It is shown below that the behaviour of $\Phi(\eta)$ as $\eta \to \infty$ plays the crucial role in separating the two cases, $\varpi_{\mathrm{cr}} < \infty$ and $\varpi_{\mathrm{cr}} = \infty$.

Consider the behaviour of the function $\chi(\rho)$ introduced in (32) and defined in interval $(1, b/a)$. It monotonically decreases from infinity, $\chi_a = \chi(1+) = \infty$, to its minimum value $\chi_b = \chi(b/a) > E_0\sqrt{1+\varepsilon} > 0$ where $\varepsilon > 0$. This means that for any selected function $\Phi(\eta)$ satisfying the conditions (8) and for any fixed set of parameters b/a

and E_0, the value of χ_b is always separated from E_0. Thus, the argument of the square root in the integrand in (31) is always positive and (31) can be rewritten, with the use of (32), in the following equivalent form

$$\varpi_{\rm cr} = \frac{2}{E_0}\int\limits_1^{b/a} \frac{\chi(\rho){\rm d}\rho}{\Phi(\xi)\rho^3}.$$

This makes it possible to estimate the value of $\varpi_{\rm cr} = \varpi_{\rm cr}\ (E_0)$ as

$$\frac{2}{E_0}\frac{a^3}{b^3}\int\limits_1^{b/a} \chi(\rho){\rm d}\rho < \varpi_{\rm cr} < \frac{2}{E_0\phi_0}\int\limits_1^{b/a} \chi(\rho){\rm d}\rho. \tag{33}$$

In other words, the value of $\varpi_{\rm cr}$ approaches infinity if and only if

$$\int\limits_1^{b/a} \chi(\rho){\rm d}\rho = \infty.$$

Thus it is completely determined by the behaviour of the function $\chi(\rho)$ in the vicinity of the point $\rho = 1$, which, in turn, depends on the behaviour of the function $\Phi(\eta)$ at infinity. The subsequent general analysis is restricted to a class of functions defined by

$$\Phi(\eta) = 1 - \phi_\infty \eta^{-\beta} + O\left(\eta^{-\beta-\delta}\right), \quad \eta \to \infty, \tag{34}$$

with $\beta > 0,\ \delta > 0$ and $\phi_\infty > 0$. Then, it follows from (32) that

$$\chi(\rho) = \frac{E_0}{2}(\rho-1)^{-1/2} + B(\rho-1)^\alpha + o\left[(\rho-1)^\alpha\right], \quad \rho \to 1, \quad \beta > 2, \quad \varpi_{\rm cr} < \infty, \tag{35}$$

$$\chi(\rho) = \frac{1}{2}\sqrt{E_0^2 + 2\phi_\infty}(\rho-1)^{-1/2} + O\left[(\rho-1)^{-1/2-\delta}\right], \quad \beta = 2, \quad \varpi_{\rm cr} < \infty, \tag{36}$$

$$\chi(\rho) = \left(\frac{\phi_\infty}{2}\right)^{1/\beta}(\rho-1)^{-1/\beta} + O\left[(\rho-1)^{-1/\max\{2,(\beta+\delta)\}}\right], \quad \rho \to 1, \quad 1 < \beta < 2, \quad \varpi_{\rm cr} < \infty. \tag{37}$$

$$\chi(\rho) = \left(\frac{\phi_\infty}{2}\right)^{1/\beta}(\rho-1)^{-1/\beta} + o\left[(\rho-1)^{-1/\beta}\right], \quad \rho \to 1, \quad 0 < \beta \le 1, \quad \varpi_{\rm cr} = \infty. \tag{38}$$

where

$$B = \begin{cases} -\frac{3}{8}E_0^3, & \beta > 4, \\ -\frac{1}{8}E_0^3\left(3 - \phi_\infty \frac{16}{E_0^4}\right), & \beta = 4, \\ \frac{1}{8}E_0^3\phi_\infty\left(\frac{2}{E_0}\right)^\beta, & 2 < \beta < 4, \end{cases} \qquad \alpha = \begin{cases} \frac{1}{2}, & \beta \ge 4, \\ (\beta-3)/2, & 2 < \beta < 4. \end{cases}$$

It is worth noting that the second term on the right-hand side of (32) plays the leading role in the case of (35) and the first term in the case of (37) and (38). The case of (36) is intermediate when both terms are important and influence the leading asymptotic term. As a result of the analysis just completed, it is possible to divide the constitutive equations of type (34) into the two groups according to the value of β:

- $\varpi_{\rm cr} < \infty$ if $1 < \beta < \infty$,
- $\varpi_{\rm cr} = \infty$ if $0 < \beta \le 1$.

A property of material models that belong to the latter group is that the regime of sticking occurs at any value of u_t because $u_t/\dot{a} < \varpi_{\rm cr} = \infty$. The regime of sticking for material models that belong to the former group is obtained for values of u_t satisfying the inequality $u_t/\dot{a} < \varpi_{\rm cr}$ (the set of parameters at which $u_t/\dot{a} = \varpi_{\rm cr}$ is tentatively excluded from consideration). In each of these cases $\tau = \tau(u_t) < 1$ (otherwise, the definition for $\varpi_{\rm cr}$ would imply

$u_t/\dot{a} = \varpi_{\rm cr}$) and $\xi_{\rm eq} < \infty$, as follows from (15) and (18). These cases will not be considered in the remainder of the present paper.

Assume that $\tau = 1$. Then, $u_t/\dot{a} > \varpi_{\rm cr}$ or $u_t/\dot{a} = \varpi_{\rm cr}$ and, therefore, material models that belong to the first group only ($1 < \beta < \infty$) should be included in analysis. The inequality corresponds to the regime of sliding, and the equality to the transition between the friction regimes. It follows from (15) that $\cos\varphi = 0$ at $\rho = 1$. Then, Eq. (18) shows that the equivalent strain rate is singular near the maximum-friction surface, $\rho = 1$, and its behaviour can be found from (35)–(37) since

$$\xi_{\rm eq}(\rho) = \xi_0 \chi(\rho), \tag{39}$$

at $\tau = 1$. Because of the range of β of interest, Eq. (38) has been excluded from (39). Substitution of (35)–(37) in (39) gives

$$\xi_{\rm eq}(\rho) = \frac{\dot{a}}{\sqrt{3}a}(\rho-1)^{-1/2} + \xi_0 B(\rho-1)^{\alpha} + o\left[(\rho-1)^{\alpha}\right], \quad \rho \to 1, \quad \beta > 2, \tag{40}$$

$$\xi_{\rm eq}(\rho) = \sqrt{\frac{1}{3}\left(\frac{\dot{a}}{a}\right)^2 + \frac{\phi_\infty \xi_0^2}{2}}(\rho-1)^{-1/2} + O\left[(\rho-1)^{-1/2-\delta}\right], \quad \rho \to 1, \quad \beta = 2, \tag{41}$$

$$\xi_{\rm eq}(\rho) = \xi_0 \left(\frac{\phi_\infty}{2}\right)^{1/\beta}(\rho-1)^{-1/\beta} + O\left[(\rho-1)^{-1/\max\{2,(\beta+\delta)\}}\right], \quad \rho \to 1, \quad 1 < \beta < 2, \tag{42}$$

where α and B involved in (40) were defined after (38).

It is indeed possible to propose functions that do not belong to the class of functions introduced in (34) but lead to singular behaviour of the equivalent strain rate in the vicinity of maximum-friction surfaces. For example,

$$\Phi(\eta) \sim 1 - \phi_\infty \exp(-\lambda\eta), \quad \eta \to \infty, \tag{43}$$

$$\Phi(\eta) \sim 1 - \frac{\phi_\infty}{\eta \log^{\gamma}\eta}, \quad \eta \to \infty, \quad \gamma > 1. \tag{44}$$

For the function introduced in (43) the asymptotic representation of the equivalent strain rate shown in (40) is valid. In the case of the function given by (44) the asymptotic representation of the equivalent strain rate is

$$\xi_{\rm eq}(\rho) = O\left[(\rho-1)^{-1}\log^{-\gamma}(\rho-1)\right], \quad \rho \to 1, \quad \gamma > 1. \tag{45}$$

6 Extreme cases

There are four extreme cases of great interest, namely (*i*) $\dot{a} \to 0$, ξ_0 is fixed and, therefore, $E_0 \to 0$, (*ii*) $\xi_0 \to \infty$, $\dot{a}$ is fixed and, therefore, $E_0 \to 0$, (*iii*) $\xi_0 \to 0$, $\dot{a}$ is fixed and, therefore, $E_0 \to \infty$, and (*iv*) $\sigma_s \to \sigma_0$, ξ_0 and $\dot{a}$ are arbitrary.

6.1 Case (i)

For $\dot{a} = 0$ the statement of the problem under consideration is reduced to that of the Couette flow studied for viscoplastic models with a saturation stress in [12]. It is of interest to compare the latter solution and the solution given in the present paper as $\dot{a} \to 0$. It has been shown in [12] that the solution for the Couette flow depends on the value of parameters b/a and ϕ_0. In particular, plastic deformation may occur in the entire interval $\rho \in (1, b/a)$ or a part of the material can be rigid. For the generalized Couette flow considered in the present paper, Eq. (32) always has a solution and therefore no rigid zone can exist. However, in the particular case of $E_0 = 0$ it immediately follows from this equation that the solution exists if and only if $b/a \le 1/\sqrt{\phi_0}$.

To illustrate the solution, the function $\Phi(\eta)$ has been chosen in the form

$$\Phi(\eta) = \frac{\phi_0 + \eta^{\beta}}{1+\eta^{\beta}}. \tag{46}$$

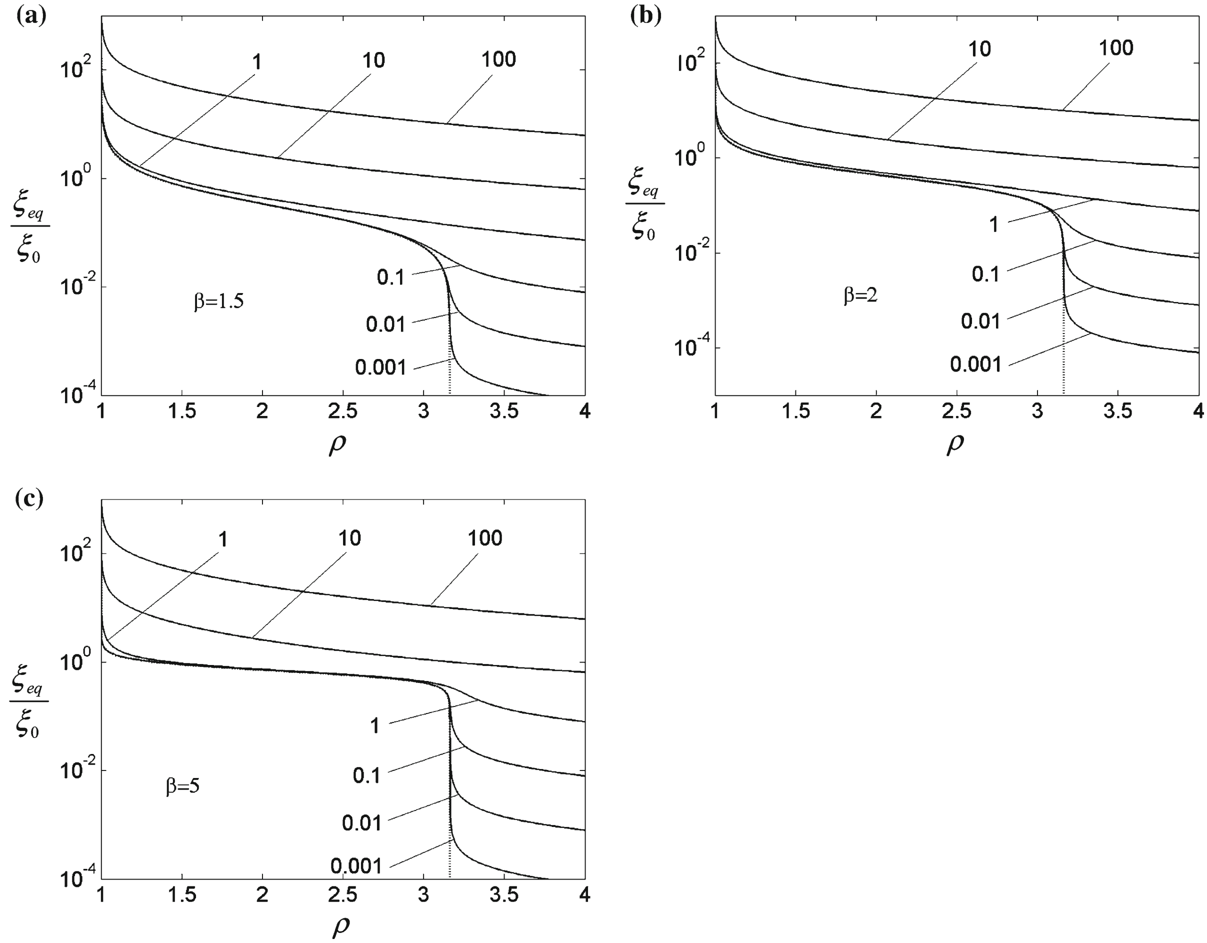

Fig. 2 Variation of the dimensionless equivalent strain rate with ρ for different values of E_0 (= 0.001, 0.01, 0.1, 1, 10, 100) and β (= 1.5, 2, 5). The *dotted line* corresponds to the solution for Couette flow

Therefore, $\phi_\infty = 1 - \phi_0$. For $b/a = 4$ and $\phi_0 = 0.1$ the radial distribution of the dimensionless equivalent strain rate, ξ_{eq}/ξ_0, is shown in Fig. 2 for different values of the parameters β and E_0. The case under consideration ($E_0 \to 0$) is illustrated by curves corresponding to smaller values of E_0.

In the case of (46) the solution to (32) for $E_0 = 0$ exists only if ρ varies in the interval $1 < \rho < 1/\sqrt{\phi_0}$, which is consistent with the solution for the Couette flow given in [12] and shown by dotted lines in Fig. 2. As $E_0 \to 0$ the solution to Eq. (32) tends uniformly to that for $E_0 = 0$ subject to the condition $\rho \leq \sqrt{\phi_0}$.

Passing in (31) to the limit as $E_0 \to 0$, one may show that

$$\varpi_{\rm cr}(E_0) = \frac{1}{E_0}\varpi_*(b/a, \beta, \phi_0) + O(1), \quad E_0 \to 0, \tag{47}$$

where $\bar{\omega}_*$ is independent of E_0. The asymptotic representation (47) immediately follows from (33) and an analysis of the leading terms in the asymptotic estimates (36) and (37). In (35), the leading term vanishes as $E_0 \to 0$ and a more accurate analysis of (32) has been carried out to prove (47). The dependence of $\varpi_{\rm cr}$ on E_0 predicted by (47) is illustrated in Fig. 3a where the result of a numerical integration in (31) is shown in logarithmic coordinates for $b/a = 4$, $\phi_0 = 0.1$ and different values of β. It is seen that all curves become almost straight lines for sufficiently small values of E_0. In Fig. 3b, the dependence of $\varpi_{\rm cr}$ on ϕ_0 is shown for $b/a = 4$, $E_0 = 0.1$ and different values of β.

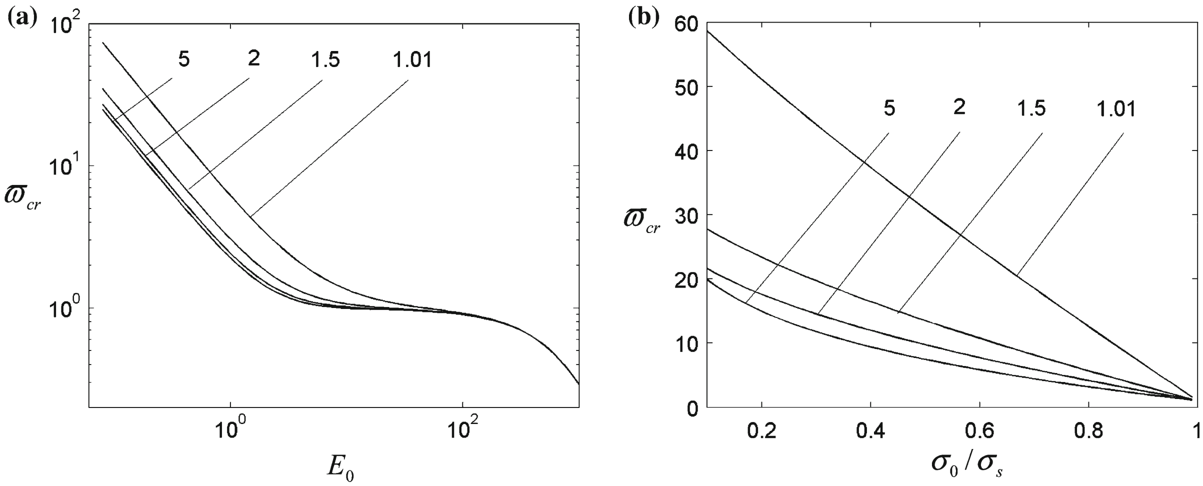

Fig. 3 Variation of ϖ_{cr} introduced in (31) with E_0 for $\phi_0 = 0.1$ (**a**) and with ϕ_0 for $E_0 = 0.1$ (**b**) for different values of β (= 1.01, 1.5, 2, 5)

It follows from (47) that $\varpi_{\mathrm{cr}} \to \infty$ as $E_0 \to 0$ not only for $0 < \beta \le 1$, as in the general case, but also for $1 < \beta < \infty$. In contrast to the general solution, the condition $\bar{\omega}_{\mathrm{cr}} \to \infty$ may not lead to the regime of sticking since $\dot{a} \to 0$ in the case under consideration and, therefore, the magnitude of $U_{\max}$ can be bounded. In particular, it follows from (47) that

$$U_{\max} = \varpi_{\mathrm{cr}}\dot{a} \sim \frac{\sqrt{3}}{2} a\xi_0\varpi_*, \quad E_0 \to 0. \tag{48}$$

Thus, the regime of sliding occurs for sufficiently large values of u_t and for any value of β in the range $1 < \beta < \infty$. This is consistent with the corresponding result for $\dot{a} = 0$ obtained in [12].

The asymptotic representations of the equivalent strain rate as $\rho \to 1$ given in (41) and (42) are valid in the case under consideration and coincide in the limit with the corresponding representations obtained in [12] for $1 < \beta \le 2$. The domain of validity of the asymptotic representation (40) vanishes as $\dot{a} \to 0$. This means that other asymptotic representations accounting for both small parameters, $\rho - 1$ and $\dot{a}$, should be constructed for $\beta > 2$. No attempt has been made to get such representations.

6.2 Case (ii)

In this case the material response given in (7) approaches the function

$$\sigma_Y = \begin{cases} \sigma_0 & \text{for } \xi_{\mathrm{eq}} \le \xi_* < \infty, \\ \sigma_s & \text{for } \xi_{\mathrm{eq}} \to \infty. \end{cases} \tag{49}$$

Therefore, for any finite value of $\xi_{\mathrm{eq}} \le \xi_* < \infty$ the viscoplastic response approaches the classical model of rigid perfectly plastic solids with $\sigma_Y = \sigma_0$. However, in the problem under consideration $\xi_{\mathrm{eq}} \to \infty$ in the vicinity of the maximum-friction surface at sliding. It is shown below that the solution obtained does not converge to the corresponding rigid perfectly plastic solution. The latter is given, for example, in [7].

The asymptotic representations (47) and (48) are valid since the only condition used for deriving these representations was $E_0 \to 0$. In particular, Eq. (48) shows that $U_{\max} \to \infty$. The latter is equivalent to the statement that the regime of sticking occurs for any value of u_t. It is not consistent with the rigid perfectly plastic solution as well as with the previous extreme case. In the present case, an effect of increasing u_t is that the equivalent strain rate increases within a significant range of ρ, whereas the rigid perfectly plastic solution requires a localization of

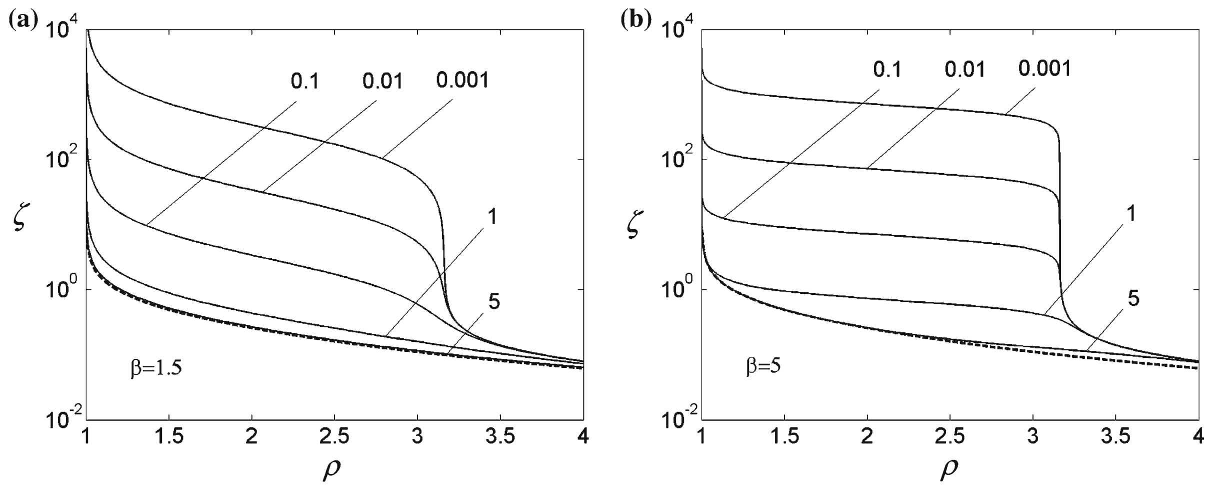

Fig. 4 Variation of the dimensionless equivalent strain rate introduced in (50) with ρ for different values of E_0 (= 0.001, 0.01, 0.1, 1, 5) and β (= 1.5, 5). The *dashed line* corresponds to the rigid perfectly plastic solution

plastic deformation in the vicinity of the maximum-friction surface. To illustrate such behaviour of the solution, it is convenient to introduce a new dimensionless equivalent strain rate in the form

$$\zeta\left(\rho\right)=\frac{\sqrt{3}a\xi_{\mathrm{eq}}}{2\dot{a}}. \tag{50}$$

Its dependence on material and process parameters is shown in Fig. 4 for the function $\Phi\left(\eta\right)$ given in (46) for $b/a=4$ and $\phi_0=0.1$ (curves corresponding to smaller values of E_0). The smaller the value of E_0, the larger the value of u_t needed to get the regime of sliding.

The difference in the interpretation of (48) for Case (*i*) and Case (*ii*) comes from the condition that ξ_0 approaches infinity in the case under consideration, but not in Case (*i*).

The asymptotic behaviour of the equivalent strain rate given in (40) to (42) is not valid in the case under consideration in the sense that for any large value of u_t there exists such a small value of E_0 that the regime of sticking occurs.

6.3 Case (iii)

In this case the material response given in (7) approaches the function

$$\sigma_Y=\begin{cases}\sigma_0 & \text{for}\ \ \xi_{\mathrm{eq}}\to 0,\\ \sigma_s & \text{for}\ \ \xi_{\mathrm{eq}}\geq\varepsilon_*>0.\end{cases} \tag{51}$$

Therefore, for any finite value of $\xi_{\mathrm{eq}}\geq\varepsilon_*$ the viscoplastic response approaches the classical model of rigid perfectly/plastic solids. However, in contrast to Case (*ii*), the tensile yield stress of this rigid perfectly plastic solid should be σ_s.

The asymptotic representations (35) and (36) show that $\chi\left(\rho\right)$ may approach infinity in a finite interval $1<\rho<1+\varepsilon$ as $E_0\to\infty$ because the coefficients of the leading asymptotic terms may tend to infinity. Therefore, it is more convenient for an asymptotic analysis of the solution to use the function $\zeta\left(\rho\right)$ introduced in (50) instead of $\chi\left(\rho\right)$. Then, it follows from (32) that

$$\rho^4=\frac{1}{\Phi^2(\zeta E_0)}+\frac{1}{\zeta^2}. \tag{52}$$

Note that any possible solution of (52) is separated from zero. In fact, $\rho^2>\zeta^{-1}$ or $\zeta>\sqrt{a/b}$ at least. However, this means that $\Phi(\zeta E_0)\to 1$ uniformly in $\rho\in(1,b/a)$ as $E_0\to\infty$ (see (51) and (7)). In other words, in the case

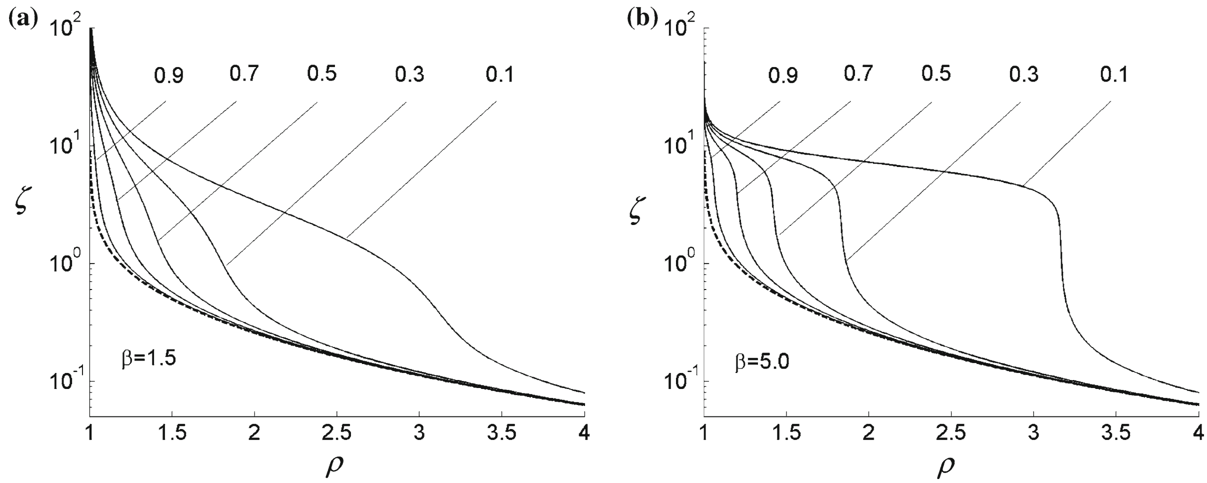

Fig. 5 Variation of the dimensionless equivalent strain rate introduced in (50) with ρ for different values of ϕ_0 (= 0.1, 0.3, 0.5, 0.7, 0.9) and β (= 1.5, 5). The *dashed line* corresponds to the rigid perfectly plastic solution

of $E_0 \to \infty$ (52) transforms to

$$\rho^4 = 1 + \frac{1}{\zeta^2}. \tag{53}$$

The solution (53) coincides with the rigid perfectly plastic solution. In particular, it follows from (53), that, as $E_0 \to \infty$, the value of $\varpi_{\rm cr}$ determining the boundary between the regimes of sticking and sliding (see the discussion after (32)) is given by $\varpi_{\rm cr}(E_0) \to \sqrt{1-(a/b)^4}$. It coincides with the rigid perfectly plastic solution as well. Such behaviour of $\varpi_{\rm cr}(E_0)$ can be seen in Fig. 3a at large values of E_0. The convergence of the viscoplastic solution to the rigid perfectly plastic solution is illustrated in Figs. 4 and 5 (in Fig. 4, curves for larger values of E_0) where the dependence of the dimensionless equivalent strain rate on ρ is shown for $b/a = 4$ and $\phi_0 = 0.1$ (Fig. 4) and $b/a = 4$ and $E_0 = 0.1$ (Fig. 5). The rigid perfectly plastic solution, $\zeta = \left(\rho^4 - 1\right)^{-1/2}$, is shown by dashed lines. Thus, each of the considered viscoplastic laws with a saturation stress results in the solution whose limit is the rigid perfectly plastic solution with $\sigma_Y = \sigma_s$.

The asymptotic representation of the equivalent strain rate given in (40) to (42) is valid if the regime of sliding occurs. Moreover, the main term in (40) coincides with that in the rigid perfectly plastic solution.

6.4 Case (iv)

It can be shown that in this case the convergence of the viscoplastic solution to the rigid perfectly plastic solution takes place regardless of other properties of the viscoplastic model. To demonstrate this fact, consider Eqs. (8) and (34) from which it follows that there exists a $\eta_* > 0$ such that

$$1 - \frac{1}{\eta_*^\beta}\phi_\infty > \phi_0 \quad \text{or} \quad \phi_\infty < (1-\phi_0)\eta_*^\beta. \tag{54}$$

If $\sigma_0 \to \sigma_s$, which is equivalent to $\phi_0 \to 1$, it is possible to obtain from (54) that $\phi_\infty \to 0$. Since the function $\Phi(\eta)$ is monotonic and $\Phi(0) = \phi_0 \to 1$ as $\sigma_0 \to \sigma_s$ simultaneously with $\Phi(\eta) \to 1$ as $\eta \to \infty$, one can conclude that $\Phi(\eta) \to 1$ uniformly as $\sigma_0 \to \sigma_s$. As a result, Eq. (53) holds true as $\sigma_0 \to \sigma_s$ or, in other words, the viscoplastic solution converges to the rigid perfectly plastic solution. To illustrate this result numerically, the distribution of the dimensionless equivalent strain rate with respect to ρ is presented in Fig. 5 for $b/a = 4$ and $E_0 = 0.1$. It is seen that the viscoplastic solution tends to the corresponding rigid perfectly plastic solution if $\phi_\infty \to 0$ (or $\sigma_0 \to \sigma_s$). The largest difference between the solutions appears near the friction boundary $\rho = 1$.

It immediately follows from (40) and (41) that for $\phi_\infty = 0$ the main terms coincide with those in the rigid perfectly plastic solution. The main term in (42) whose behaviour differs from that in the rigid perfectly plastic solution vanishes as $\phi_\infty \to 0$ and, therefore, a new asymptotic representation of the equivalent strain rate should be found in this case.

It is hypothesized that the convergence of the viscoplastic solutions to the corresponding rigid perfectly plastic solutions as $\phi_\infty \to 0$ (or $\sigma_0 \to \sigma_s$) is a general property of the viscoplastic model under consideration, i.e., it is independent of the specific boundary-value problem.

7 Concluding remarks

It has been shown that the equivalent strain rate may approach infinity in the vicinity of maximum-friction surfaces in viscoplastic solids with a saturation stress. Moreover, the asymptotic behaviour of this quantity can be the same as in rigid perfectly plastic solutions. This may have a great effect of the solvability of classical problems of plasticity when viscoplastic models are adopted. In particular, a generalization of the famous Prandtl problem, compression of a block between two parallel, rough plates, on viscoplastic materials proposed in [17] is not valid in the case of the maximum-friction law. It has been demonstrated in [18,19] that an appropriate generalization can be obtained if viscoplastic solids with a saturation stress are adopted. Moreover, a number of results related to the asymptotic behaviour of the equivalent strain rate in the vicinity of the maximum-friction surface obtained in that work coincide with the corresponding results given in the present paper.

Several extreme cases have been investigated. In particular, it has been shown that the viscoplastic solution obtained here converges to the corresponding rigid perfectly plastic solution in Case (*iii*) and Case (*iv*) but not in Case (*ii*). This result can be considered as a first step towards the development of a unified theory of plasticity that would cover a wide range of temperatures and strain rates and would include conventional theories as particular cases. The issue of convergence is of importance for developing such a unified theory since it is obvious that the area of applicability of this or that constitutive model is rather conditional on the specific area of application. To this end, it is of interest to study the viscoplastic solution to a problem that involves both a maximum-friction surface and a rigid zone. It is expected that in this case the viscoplastic solution satisfying the conditions formulated in Case (*iii*) does not converge to the rigid perfectly plastic solution. This will be the subject of a subsequent investigation.

The asymptotic behaviour found can be of importance in the area of computational mechanics. For, viscoplastic models with a saturation stress are used in applications (for example, [20]) and friction elements are often considered separately in a finite-element analysis of metal-forming processes, even if the solution is not singular (see, for example, [21]). Obviously, the singular asymptotic representation of the solution found in the present paper can be accounted for in such elements that should increase the accuracy of numerical results in the vicinity of frictional interfaces. Nevertheless, infinite strain rates frequently occur in rigid plastic solutions and it is necessary to somehow relate such theoretical solutions and practical applications.

The specific form of the asymptotic representation of the solution depends on the asymptotic behaviour of the yield stress as the equivalent strain rate approaches infinity and thus is beyond the range of applicability of any direct experiment. Therefore, the coincidence of qualitative behaviour of mathematical solutions is a good argument to choose this or that model. Using this argument for the constitutive law defined by (34), it is possible to recommend to adopt $\beta \geq 2$ at very high strain rates for materials whose behaviour is adequately described by a rigid perfectly plastic model under a wide range of strain rates. It is worth noting here that a small effect of viscosity is actually revealed in such materials at very high strain rates [22].

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